J. Fluid Mech. (1977), *vol.* 79, *part* 2, *pp.* 361–377
Printed in Great Britain

Hypersonic flow with attached shock waves over plane delta wings

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(Received 26 August 1975 and in revised form 2 September 1976)

A new form is given for the general solution to the thin-shock-layer equations for the flow over a nearly plane delta wing. Using this, the solution described conjecturally by Hayes & Probstein for symmetrical flow with attached shock waves over a plane delta wing is realized numerically. The flow construction devised for this purpose is applied also to yawed flows. The solutions obtained are found to agree moderately well with the results of numerical calculations from the full equations, but contain a number of anomalous features characteristic of the thin-shock-layer approximation.

1. Introduction

In this paper, we give a solution to the thin-shock-layer equations for hypersonic flow which satisfies the boundary conditions appropriate to the flow on the compression side of a plane delta wing when the shock wave is attached to the leading edges.

The problem of calculating such flows is of some importance in aeronautics, in connexion, for example, with studies of lifting bodies at very high speeds. We shall begin by reviewing previous work in applying the thin-shock-layer approximation to it. As a point of departure we take Messiter's (1963) paper, in which the approximation in the form in which we shall use it was first formulated, and in which a solution to the problem which has been used and extended by many subsequent investigators was first given.

Messiter's development of the thin-shock-layer approximation as an improvement on Newtonian theory resulted from the following considerations. The shock wave on a nearly plane body at incidence to a hypersonic stream will be close to the body, at least on the windward side; the shock layer, between the shock wave and the body, will be $O(\epsilon)$ in thickness, where ϵ, the ratio of the density upstream of the shock wave to that downstream, becomes vanishingly small in the Newtonian limit $M \to \infty$, $\gamma \to 1$. The Mach angle within the shock layer is $O(\epsilon^{\frac{1}{2}})$ in the same limit. Thus Messiter dilated the scales for the co-ordinate and velocity component normal the body surface by division by ϵ and dilated the spanwise co-ordinate and velocity component by division by $\epsilon^{\frac{1}{2}}$, so that the structure of the shock layer, including the central region of conically subsonic flow in the case of the delta wing, could be investigated in the limit as $\epsilon \to 0$.

Messiter's calculations were restricted to cases when the shock wave is detached from the leading edges of the wing; he remarked, without going into a detailed discussion, that when the shock wave is attached "a difficulty arises in matching the uniform-flow solution (near the leading edge) to the non-uniform flow over the inboard region".

The difficulty referred to consists of this: notwithstanding that the scaling adopted is motivated by the wish to preserve the extent of the non-uniform region within the Mach cone of the wing vertex in the limit $\epsilon \to 0$, and despite the fact that the full equations of motion are elliptic in character in this region (or, more precisely, possess one (repeated) real and two complex families of characteristics there), the approximate equations which remain after the passage to the limit are hyperbolic throughout. Thus, for the attached-shock flow on a plane delta wing, the boundary conditions on the shock and at the wing and the fact that the shock wave is attached to the leading edge appear to provide complete Cauchy data for these equations, so that the requirement (in the case of zero yaw) that the flow be symmetrical about the central plane appears to overdetermine the problem.

This difficulty was discussed in greater detail by Hayes & Probstein (1966), who gave a conjectural description of the solution which they believed would apply for a plane wing with an attached shock. A distinctive feature of this solution was an infinite sequence of discontinuities in shock slope. Their discussion did not however provide a prescription for the numerical realization of this solution, and none has appeared subsequently. The problem of attached-shock flows in the thin-shock-layer approximation was attacked rather differently by Squire (1967), who avoided the matching problem by relaxing the wing boundary condition over a relatively small interval of the span in the non-uniform inboard region, so that a smooth solution could be constructed, which was in fact appropriate to a wing which was planar over most of its span and which deviated from planarity by relatively little over the remainder. Later Roe (1970) presented an even simpler solution along the same lines.

The first attached-shock solution that did not compromise the wing boundary condition to overcome the matching difficulty was given by Woods (1970). This was particularly simple, comprising as it did a mosaic of regions of constant flow separated by permissible discontinuities which are initiated by a jump in shock slope, positioned such that the resulting flow is symmetric about the centre-plane. Notwithstanding the somewhat counter-intuitive nature of this solution, it did reproduce some of the features predicted by Hayes & Probstein, and predicted pressure distributions on plane delta wings, both in symmetrical and in yawed flow, moderately well. However, it violated a principle proposed by those authors for shock slope discontinuities in thin-shock-layer flow, namely that the jump must always represent a transition from super- to subcritical flow. In Woods's solution, the initial jump is in fact from sub- to supercritical flow. As remarked by Professor Hayes (1971, private communication), commenting on the two distinct solutions given in Woods's paper for the flow at the tip of a rectangular wing: "If sub- to super-jumps are permitted, non-uniquenesses appear generously through all thin-shock-layer theories." The solutions given in the present paper do not violate this principle however, and possess all the features predicted for the solution sketched in Hayes & Probstein's monograph.

Other methods have been used to calculate attached-shock flows on delta wings. Direct numerical computations, using finite-difference approximations to the full equations, have been made by Babaev (1963*a*, *b*), Voskresenskii (1968) and Klunker, South & Davis (1971). Generally speaking, these calculations have been made for cases which could hardly be regarded as hypersonic, so that it has not been possible for us to compare systematically calculations using our solution with these 'exact' solutions under conditions in which thin-shock-layer theory might be expected to give reasonable results. However, some comparisons are made in § 4 of this paper.

An alternative approximate method for calculating hypersonic flows over delta wings, initiated by Malmuth (1966) and extended by Hui (1971, 1973) and Malmuth (1973), has been to treat the non-uniform, conically subsonic region of flow, near the centre-line, as a small perturbation to the uniform flow behind the oblique shock which a plane wedge at the same incidence as the delta wing would generate. Malmuth's work is based on the hypersonic small-disturbance theory, but that of Hui (in which, in addition, a simple co-ordinate stretching is employed to ensure that the Mach cone from the wing vertex is located exactly) does not rely on this approximation, and is applicable down to moderate supersonic Mach numbers. Again, we have performed calculations for comparison with some of those made using this linearized approach, and these are presented below in § 4.

In the next section we outline Messiter's derivation of the thin-shock-layer equations for conical flow, present the appropriate boundary conditions for a nearly plane delta wing, and obtain a general solution to these in a new form, differing from but equivalent to that given earlier by Messiter. In the following section we describe a procedure for calculating the flow with attached shock waves on a delta wing; this is found to be applicable to plane wings in symmetric and yawed flow. Some calculations based on this procedure are presented in § 4, and are compared with the results of other theoretical methods.

Finally, in § 5 some concluding remarks are made on the applicability and accuracy of thin-shock-layer theory for attached-shock flows.

2. The approximate equations and their general solution

Figure 1 shows the position of the wing and the direction of the undisturbed incident flow relative to a set of Cartesian axes with origin at the wing vertex. We use new independent variables

$$x = \bar{x}/l, \quad y = \bar{y}/\bar{x}\epsilon\tan\alpha, \quad z = \bar{z}/\bar{x}\epsilon^{\frac{1}{2}}\tan\alpha, \tag{1}$$

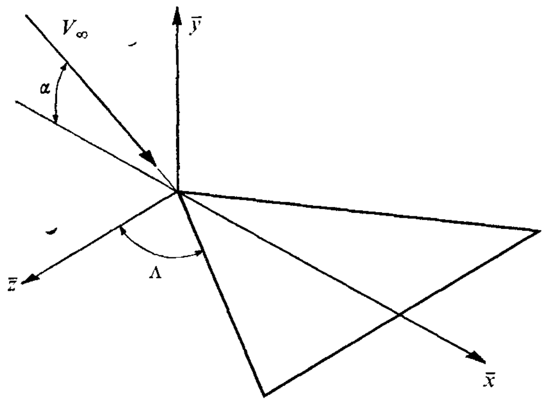


FIGURE 1. Co-ordinate system.

and, in terms of these, write the dependent variables as

$$\left.\begin{aligned}\bar{u}(\bar{x},\bar{y},\bar{z}) &= V_\infty(\cos\alpha+\epsilon\sin\alpha\tan\alpha u(y,z)),\\ \bar{v}(\bar{x},\bar{y},\bar{z}) &= V_\infty\,\epsilon\sin\alpha v(y,z),\\ \bar{w}(\bar{x},\bar{y},\bar{z}) &= V_\infty\,\epsilon^{\frac{1}{2}}\sin\alpha w(y,z),\\ \bar{p}(\bar{x},\bar{y},\bar{z}) &= \bar{\rho}_\infty\,V_\infty^2\sin^2\alpha(1+\epsilon p(y,z))+p_\infty,\\ \bar{\rho}(\bar{x},\bar{y},\bar{z}) &= \bar{\rho}_\infty(\epsilon^{-1}+\rho(y,z)).\end{aligned}\right\} \tag{2}$$

Here the barred symbols $\bar{x},\ldots,\bar{u},\ldots,$ denote unscaled, dimensional quantities, while the unbarred symbols denote the corresponding scaled, dimensionless quantities. The small parameter ϵ is taken to be the density ratio across a shock wave lying in the $\bar{x}$, $\bar{z}$ plane:

$$\epsilon = \frac{\gamma-1}{\gamma+1}\left\{1+\frac{2}{(\gamma-1)\,M_\infty^2\sin^2\alpha}\right\}. \tag{3}$$

The passage to the limit $\epsilon\to 0$ is understood to be effected by the double (Newtonian) limit $M_\infty\sin\alpha\to\infty$, $\gamma\to 1$, so that $(\gamma-1)\,M_\infty^2\sin^2\alpha$ remains a constant greater than zero.

When expressed in terms of the scaled variables, and after this passage to the limit, the full equations of motion for the steady flow of an inviscid perfect gas reduce to

$$\partial v/\partial y+\partial w/\partial z = 0, \tag{4a}$$

$$(v-y)\frac{\partial u}{\partial y}+(w-z)\frac{\partial u}{\partial z} = 0, \tag{4b}$$

$$(v-y)\frac{\partial v}{\partial y}+(w-z)\frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y}, \tag{4c}$$

$$(v-y)\frac{\partial w}{\partial y}+(w-z)\frac{\partial w}{\partial z} = 0. \tag{4d}$$

The shock wave is assumed to be conical, and given in scaled co-ordinates by

$$y = y_s(z). \tag{5}$$

The Rankine–Hugoniot equations (again in the limit $\epsilon \to 0$) then give the following boundary conditions for u, v, w and p on this curve:

$$u_s = u(y_s, z) = -y_s + zy_s', \tag{6a}$$

$$v_s = v(y_s, z) = y_s - zy_s' - y_s'^2 - 1, \tag{6b}$$

$$w_s = w(y_s, z) = -y_s', \tag{6c}$$

$$p_s = p(y_s, z) = 2y_s - 2zy_s' - y_s'^2 - 1. \tag{6d}$$

Here we have written y_s' for dy_s/dz.

The wing surface is, from the conditions of the problem, conical, and we write it as

$$y = y_b(z). \tag{7}$$

The flow tangency condition on this surface yields two alternative conditions.

(*a*) If the flow at the wing surface is not radial, then the slope of the conical projection of a sheet of streamlines which originates from a ray on the shock, which we shall refer to as a 'conical streamline', must coincide with the slope of the wing, i.e.

$$(v(y_b, z) - y_b)/(w(y_b, z) - z) = y_b'. \tag{8a}$$

(*b*) If, however, the flow is radial, then the wing surface must be the locus of singular points of the equation for conical streamlines,

$$dy/dx = (v-y)/(w-z),$$

so that we have

$$v(y_b, z) = y_b, \quad w(y_b, z) = z. \tag{8b}$$

Other boundary conditions which arise (for example, on the plane of symmetry of the wing or its leading edges) will be discussed with the solutions to which they apply.

We now give a general solution to these equations and boundary conditions which differs somewhat from that originally given by Messiter, and which we shall find lends itself more readily to the calculation of the Hayes & Probstein solution and indeed to the calculation of flows with attached shock waves in general.

As has been pointed out in previous work, the solution to the problem as a whole depends on the solution of (4*a*, *d*), which form a pair of quasi-linear hyperbolic first-order equations. The characteristics for this pair are the conical streamlines and lines z = constant. We note from (4*d*) that w is constant along the conical streamlines, and so a transformation into characteristic co-ordinates is effected by transforming (4*a*, *d*) into a pair of equations for v and y in terms of z and w as independent co-ordinates. These are

$$\partial v/\partial w - \partial y/\partial z = 0, \tag{9}$$

$$\partial y/\partial z = (v-y)/(w-z), \tag{10}$$

and are now linear. If a function $\phi(w, z)$ such that

$$v = \partial\phi/\partial z, \quad y = \partial\phi/\partial w \tag{11}$$

is introduced, then (9) is satisfied identically and (10) becomes

$$\frac{\partial^2\phi}{\partial z\,\partial w}+\frac{1}{(w-z)}\left(\frac{\partial\phi}{\partial w}-\frac{\partial\phi}{\partial z}\right)=0. \tag{12}$$

This is an example of the Euler–Poisson–Darboux equation (see, for example, von Mises (1958, pp. 165–167) for a rather complete discussion, with references). In fact, if we substitute $x=-w$ for w, (12) becomes

$$\frac{\partial^2\phi}{\partial x\,\partial z}-\frac{1}{(x+z)}\left(\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial z}\right)=0, \tag{12a}$$

which may be recognized as having precisely the same form as Riemann's equation for the unsteady flow in one dimension of a perfect gas with adiabatic index $\gamma=\frac{1}{3}$. Indeed, it belongs to the special class of cases (when γ is of the form $(2N+1)/(2N-1)$, N an integer) for which there exist elementary explicit solutions; for (12a) the solution is

$$\phi(x,z)=F(x)+G(z)-\tfrac{1}{2}(x+z)\,(F'(x)+G'(z)), \tag{13}$$

where F and G are arbitrary twice-differentiable functions. Reverting to the dependent variables v and y, in terms of z and w, we thus have

$$v(z,w)=f(z)+g(w)+(w-z)f'(z), \tag{14a}$$

$$y(z,w)=f(z)+g(w)+(w-z)\,g'(w), \tag{14b}$$

where f and g are arbitrary differentiable functions related to F and G.

These arbitrary functions must be determined by the boundary conditions. We first consider those at the shock wave, $y=y_s(z)$. By differentiating (14a) along this curve and using (6b, c) we obtain a relation connecting the function f with the distribution of sidewash $w_s(z)$ on the shock:

$$f''(z)=-w_s'(z)\,[1-(w_s(z)-z)^{-2}]. \tag{15}$$

A similar relation for the function g is obtained by differentiating (14b); here it is convenient to regard the shock wave as being defined by the inverse of the relation $w=w_s(z)$, say $z=z_s(w)$; then we obtain

$$g''(w)=-z_s'(w)/(w-z_s(w))^2. \tag{16}$$

In these formulae, a prime again denotes differentiation, so that, for example, $g''(w)=d^2g/dw^2$.

Equations (15) and (16) determine f and g only to within arbitrary linear terms. Further, linear terms in these functions are interchangeable; from (14) it is clear that if a flow is specified by $f=f_1(z)$ and $g=g_1(w)$, then it is equally specified by

$$\left.\begin{aligned}f_2(z)&=f_1(z)+Az+B,\\ g_2(w)&=g_1(w)-Aw-B,\end{aligned}\right\} \tag{17}$$

where A and B are any constants.

The boundary condition (8a) at the wing $y=y_b(z)$ for the case of non-radial flow yields

$$f'(z)+g'(w)=y_b'(z), \tag{18}$$

so that on the wing surface we must have $w =$ constant (which of course is a direct consequence of ($4d$) since in this case the wing surface is a conical streamline). In the alternative case of radial flow on the surface, we have from ($8b$) and ($14b$)

$$y_b(z) = y(z, z) = f(z) + g(z). \tag{19}$$

In the latter case, the functions f and g can be eliminated from (15), (16) and (19) to give the functional–differential equation for w_s first derived by Messiter (1963, equation 3.21):

$$w'[w(z)]\, w'(z) \left\{\frac{1}{(w[w(z)] - w(z))^2} - 1\right\} = \frac{1}{(w(z) - z)^2} + w'(z)\, y_b''[w(z)]. \tag{20}$$

This equation has been widely used to calculate flows with detached shock waves. However, it will be shown in what follows that the new formulation of the solution in terms of the functions f and g, while in principle equivalent to Messiter's original solution, greatly clarifies problems arising in the calculation of attached-shock flows.

Once ($4a, d$) have been solved, the calculation of the pressure distribution is a matter of quadrature. From ($4c$) and ($14a$) we have

$$\left.\frac{\partial p}{\partial y}\right|_{z=\text{const}} = -(w - z)^2 f''(z),$$

which, using ($14b$), we can integrate to obtain

$$p(z, w) = p_s(z) + f''(z) \int_{w_s(z)}^{w} g''(t)\, (t - z)^2\, dt. \tag{21}$$

Finally, we note that the transformation used in this section cannot be used in a region of constant w, for there the Jacobian vanishes. Further, in flows in which $\partial w/\partial y$ changes sign, it will not be one-to-one. This will be the case along streamlines (lines $w =$ constant) which originate at points on the shock at which $w'(z)$ changes sign, and from (15) we see that such points occur either where $f''(z)$ changes sign or (in a more dramatic fashion) where $(w_s - z)^2 - 1$ changes sign. In either event it should be possible to divide the flow field into a number of regions in each of which the transformation and the resulting solution are valid. This is certainly the case in the calculations presented below.

3. Construction of the Hayes & Probstein solution

The flow in the immediate neighbourhood of the leading edge of a plane delta wing at $y = 0$, $z = \Omega$ may be expected to be uniform, corresponding to that behind a skewed oblique plane shock wave attached to the leading edge. Using the shock equations ($6a$–d), this flow is found to be

$$v = 0, \quad w = w_0, \quad p = 2w_0\Omega - w_0^2 - 1, \quad y_s = w_0(\Omega - z), \tag{22}$$

where

$$w_0 = \tfrac{1}{2}[\Omega - (\Omega^2 - 4)^{\frac{1}{2}}]. \tag{23}$$

From (23), it is clear that the shock will not be attached for $\Omega < 2$. On the wing surface inboard of the leading edge, for $\Omega \geqslant z \geqslant w_0$, boundary condition ($a$)

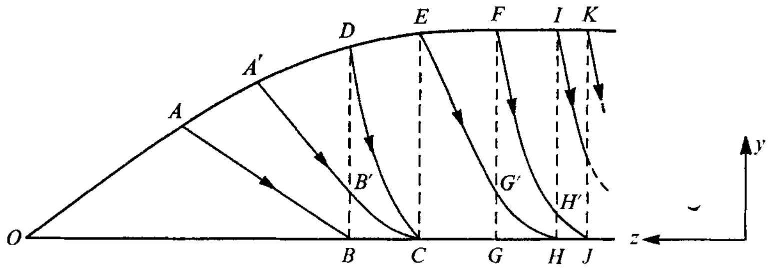


FIGURE 2. Flow-field construction.

applies. Thus, from (18), we have $f''(z) = 0$ in this interval, and subsequently, from (15), either

$$w_s'(z) = 0 \tag{24a}$$

or

$$w_s(z) - z = \pm 1. \tag{24b}$$

The first case corresponds to a plane shock wave, and the second to the 'simple wave' solution previously described by one of us (Woods 1970). The existence of two distinct flows each satisfying the same boundary conditions in this interval, together with the fact that they can be patched together with a degree of arbitrariness, provides the key to the flow construction which we now propose, and which is illustrated by figure 2. We suppose that the plane attached shock wave, with its associated uniform flow, continues inboard from the leading edge O to the point A ($z = 1 + w_0$, $y = 1 - w_0$), at which the shock wave corresponding to the simple wave solution [equation (24*b*)] has the same slope. The simple wave solution is joined onto the uniform flow at this point, and along the streamline AB', and is continued until it is terminated arbitrarily at the point A' (at $z = 1 + w_0 - \Delta$), at which the plane shock solution (24*a*) is reimposed, with $w_s = w_0 - \Delta$. This continues until the point D ($z = w_0$) is reached; here the type (*a*) boundary condition on the wing surface ceases to apply, and we no longer have $f(z) = 0$. Thus, in the flow field between the leading edge and the line BD (at $z = w_0$) we have three separate regions:

$(\mathrm{i})_a$ In OAB we have uniform flow, with

$$v = 0, \quad w = w_0, \tag{25}$$

and

$$y_s = w_0(\Omega - z). \tag{26}$$

$(\mathrm{ii})_a$ In $AA'B'B$ we have simple wave flow, with

$$f(z) = 0, \quad g(w) = \tfrac{1}{2}((w_0 - 1)^2 - (w - 1)^2), \tag{27}$$

while on AA'

$$y_s(z) = \tfrac{1}{2}[(1 - w_0)^2 - (z - 1)^2 + 1]. \tag{28}$$

We remark that the fact that one of the arbitrary functions in the general solution, $f(z)$, is constant adds point to the application of the term 'simple wave' to the flow in this region.

$(\mathrm{iii})_a$ In $A'B'D$ we have uniform flow, with

$$v = -\Delta(1 - w_0) - \tfrac{1}{2}\Delta^2, \quad w = w_0 - \Delta, \tag{29}$$

and the shock wave $A'D$ is given by

$$y_s = 1+w_0^2-\Delta(1+w_0)+\tfrac{1}{2}\Delta^2-z(w_0-\Delta). \tag{30}$$

The solution described so far provides the boundary conditions required to determine the solution in the region $BCED$, which is also divided into three parts.

(i)$_b$ In BCB', the general solution (14) may be used. The function $g(w)$ is given on the boundary BB' from the previous solution

$$g(w) = \tfrac{1}{2}[(w_0-1)^2-(w-1)^2],$$

and since now boundary condition (b) applies on the wing surface BC, from (19) we have

$$f(z) = \tfrac{1}{2}[(z-1)^2-(w_0-1)^2]. \tag{31}$$

This solution provides the boundary value

$$v(z, w_0-\Delta) = -\tfrac{1}{2}(z-w_0+\Delta)^2 \tag{32}$$

on the line $B'C$, given by

$$y(z, w_0-\Delta) = \tfrac{1}{2}(z-w_0+\Delta)^2. \tag{33}$$

(ii)$_b$ In $B'CD$, w is constant $(=w_0-\Delta)$, and the general solution is not applicable; however, from (4a) and (32) we have immediately

$$v = v(z) = -\tfrac{1}{2}(z-w_0+\Delta)^2, \tag{34}$$

and this gives sufficient boundary data for the adjacent region DCE.

(iii)$_b$ In DCE, the shock boundary condition (15) on DE may be regarded as an ordinary differential equation for $w_s(z)$ on the shock:

$$\frac{dw_s}{dz} = -f''(z)\frac{(w_s-z)^2}{(w_s-z)^2-1}, \tag{35}$$

since it turns out from elementary consideration of streamline geometry that $f''(z)$ in this region is precisely that in region $BB'C$, and is therefore known. The initial condition for w_s at D $(z = w_0)$ is not however its value $w_0-\Delta$ on $A'D$; since $f'(z)$ changes discontinuously at $z = w_0$, there is a jump in v on this line, and, consequently, a jump in shock slope at D. The shock slope just inboard of D is calculated from the slope of the streamline DC at this point; this is

$$\Delta-1+\Delta^{-1} = S,$$

say, and from the shock boundary conditions (6b, c), we have

$$w_s(w_0-) = 2(Sw_0-1)/\{S-w_0+[(S+w_0)^2-4]^{\frac{1}{2}}\}. \tag{36}$$

Integration of (35) with the initial value (36) then determines $w_s(z)$ on the shock wave DE, and also, by a further integration of (6c), fixes the position of this segment of the shock wave. These integrations are terminated at $z = w_0-\Delta$, inboard of which point $f''(z)$ is again zero, for across the jump in shock slope at AD the value of w_s is also discontinuously reduced, i.e. $w_s(w_0-) < w_0-\Delta$, and so between the points C $(z = w-\Delta)$ and G $(z = w_s(w_0-))$ on the wing surface, the type (a) boundary condition again applies. Because of this, $f''(z)$ is in fact zero throughout the region $CGFE$, and the shock wave segment EF is therefore straight, but because of the discontinuous change in $f'(z)$ there is at E another jump in shock slope. The region $CEG'G$ is thus divided into two parts.

$(\mathrm{i})_c$ In $CEG'G$, knowing that $f''(z) = 0$, we may, from (17), set $f(z) = 0$, i.e. transfer all linear terms to $g(w)$. The boundary conditions $v = y = 0$ on the wing surface ($w = w_s(w_0-)$) then give $g(w_0-) = g'(w_0-) = 0$, and we have finally

$$f(z) = 0, \quad g(w) = \int_{w_s(w_0-)}^{w} du \int_{w_s(w_0-)}^{u} g''(v)\, dv,$$

where $g''(w)$ has been determined, through (16), from the function $w_s(z)$ obtained by integration on DE, as described in $(\mathrm{iii})_b$ above.

$(\mathrm{ii})_c$ In EFG', the flow is uniform, and is given by

$$w = w_s(w_0-\Delta-),$$
$$y = y_s(w_0-\Delta)+(w-\Delta)\, w_s(w_0-\Delta-)-[w_s(w_0-\Delta-)]^2-1.$$

As we have noted, there is a jump in shock slope at E, and the value of w_s just inboard of it, $w_s(w_0-\Delta-)$, is calculated from the slope of the dividing streamline EG' through an expression analogous to (36). From the solution given in $(\mathrm{i})_c$ above, this slope is

$$\int_{w_s(w_0-)}^{w_s(w_0-\Delta+)} g''(v)\, dv.$$

On the line $GG'F$ (at $z = w_s(w_0-)$) boundary conditions similar to those on $BB'D$ now apply, so that the calculation of the flow in the region $GG'FIH$ follows the steps outlined in $(\mathrm{i})_b$, $(\mathrm{ii})_b$ and $(\mathrm{iii})_c$ above. There is however this difference: because of the discontinuity in shock slope at E, the line $EG'H$ is a shear discontinuity, separating flows characterized by different values of w. Thus, as we indicate in figure 2, the region $G'FH'JHG'$ of constant w extends inboard beyond H, the inboard terminus of the next interval GH on which boundary condition (b) applies. As a result, there is no jump in shock slope at the point I on the shock (where the numerical integration of (35) ends in this cycle) but there is a jump at the point K.

The construction is then continued inboard, in a sequence of cycles in which a segment of curved shock is generated by numerical integration (reflecting the applicability of boundary condition (b) in that interval of z) and is followed by a number of straight shock segments (reflecting the applicability of boundary condition (a)) separated by discontinuities in slope. Unhappily (from the computational point of view) each successive cycle differs from the previous one; in fact, the number of jumps in shock slope between each segment of curved shock increases at the rate of one every two cycles. In addition, the intervals over which numerical integration is performed are found to become progressively shorter; and, since each successive integration employs the results in interpolated form of the previous integration, the calculation soon becomes ill conditioned. For this reason, we cannot assert without reservation that the flow construction converges in the sense that, for a given initial choice of the determining parameter Δ, the construction outlined above proceeds such that by continuing it for a sufficiently large number of cycles we can reduce $|z-w_s(z)|$ to as small a value as we please. We observe, however, that in the first few cycles $|z-w_s(z)|$ has been found, for all values of Δ and Ω, to diminish markedly after each cycle. In the following section we shall briefly describe an approximation to which we have resorted after the first three cycles of integration, and which is found (numerically) to converge. So in what follows we shall speak of 'convergence' without further qualification.

The problem of constructing a flow field for a delta wing with attached shock waves is thus reduced to that of determining the value(s) of the parameter Δ for which the boundary conditions in the non-uniform central region are satisfied. For the symmetrical flow case (at zero yaw), for example, Δ must be such that the flow construction converges to $|z-w_s| = z = 0$.

Finally in this section, we outline the procedure for calculating the surface pressure at a given point on the span (a given value of z), once the flow field has been determined. We shall consider a point lying between B and C in figure 2. First, the pressure at the shock wave is calculated from (6d). Then the integral in (21) is evaluated, at the appropriate value of z, for the region between the shock wave and the streamline DC, and for the region between the streamline $B'C$ and the wing surface. There remains the pressure difference between these two streamlines. In the region $B'CD$, as is pointed out above, the general solution is inapplicable, and a particular solution w = constant, $v = v(z)$ holds. The pressure is thus [from (4c)] a linear function of y at fixed z in this region, and we have, in summary,

$$p(z) = p_s(z) - g''(z)\left\{\int_{w_s(z)}^{x_{DC}} g''(t)\,(t-z)^3\,dt + (w_{B'C}-z)^2\,(z-z_C)\,\frac{y_D-y_{B'}}{z_B-z_C} + \int_{w_{B'C}}^{z} g''(t)\,(t-z)^3\,dt\right\}.$$

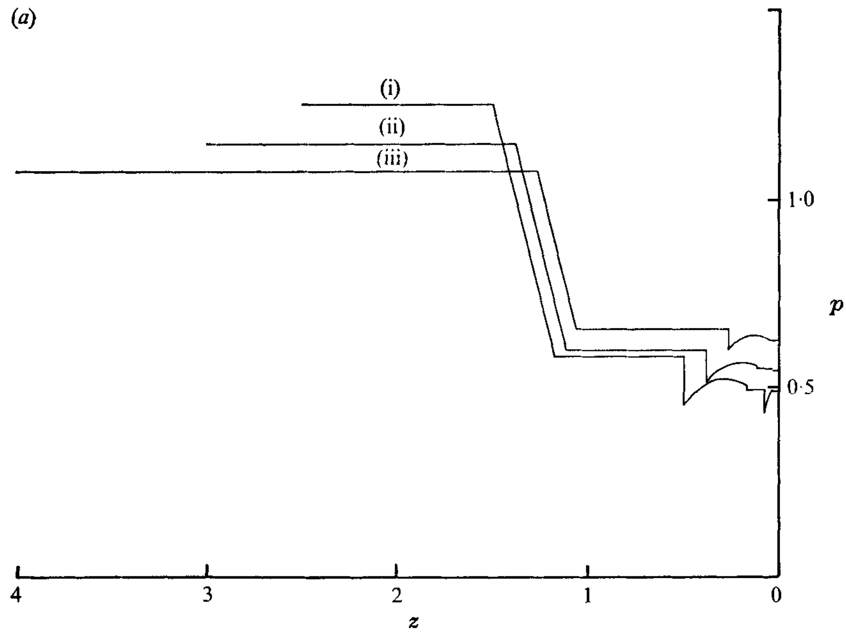
Here the suffixes refer to points and streamlines in figure 2.

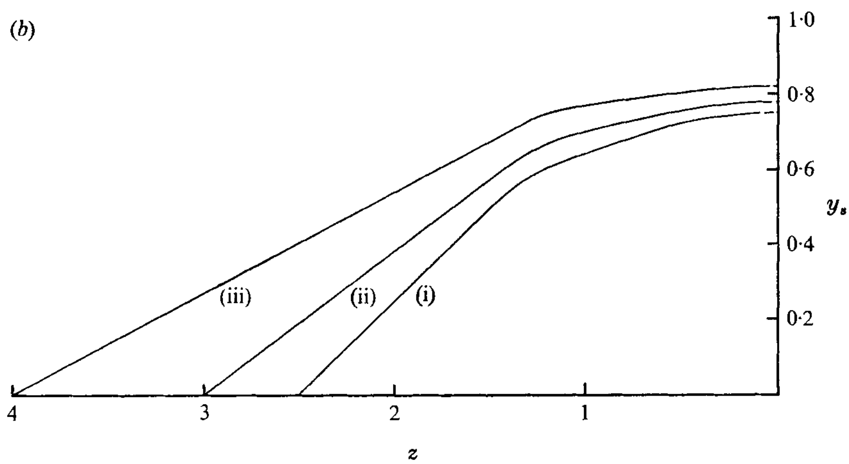
The surface pressure at a point at which $f''(z) = 0$ (that is, at which boundary condition (a) applies) is, from (21), identical with the shock wave pressure at the same value of z. Thus the jumps in the pressure distribution which are evident in figures 3–6 are due as much to effects within the flow field as to the discontinuous changes in shock slope.

4. Calculations

The flow construction described in the previous section has been programmed for the Canterbury University Burroughs B6718 computer. Each interval over which (35) had to be integrated was divided into five subintervals, and the values of $w(z)$ and $w'(z)$ at the ends of each subinterval converted into a cubic spline approximation for z as a function of w for the corresponding interval in w, the coefficients of which were stored for subsequent calculation of $g''(w)$, using (16).

As has been pointed out above, the flow construction, if convergent, involves (in principle) an infinite number of steps. We have found that, on the one hand, only three cycles of calculation as described in § 3 are required to reduce $|z-w_s|$





FIGURES 3 (a, b). For legend see next page.

to a small fraction of its original value (0·5 % for $\Omega = 2{\cdot}25$, and less for greater values of Ω), while on the other hand, evidence of numerical instability has appeared in the fourth cycles in some cases. The latter could perhaps have been delayed by refining the program, but it is evident that, as the intervals over which integration and interpolation become smaller, the value of $g''(w)$ becomes larger, and it becomes increasingly difficult to avoid numerical ill-conditioning. We have therefore adopted the following approximation after the third cycle. Referring to figure 2, we know that after the third numerical integration we have boundary conditions on the line $IH'H$, i.e. a known distribution of w on IH'

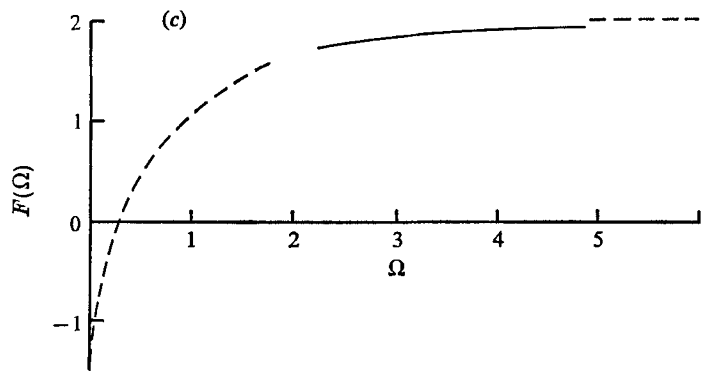


FIGURE 3. (*a*) Pressure distributions and (*b*) shock profiles for (i) $\Omega = 2{\cdot}5$, (ii) $\Omega = 3{\cdot}0$ and (iii) $\Omega = 4{\cdot}0$. (*c*) Correction function for normal-force coefficient.

and a known constant value of w on $H'H$. We thereafter replace the distribution of w on IH' by a constant value, equal to the mean of $w_s(z)$ taken over the segment FI of the shock. This is, from (6*c*), simply equal to $(y_I - y_F)/(z_F - z_I)$. The subsequent calculation involves only piecewise-constant flow regions and straight segments of shock, and proceeds by purely algebraic computation in a manner similar to the flow construction described by Woods (1970), in which the whole flow field was in fact made up of such constant regions. This approximation to the actual flow converges in the sense we have spoken of in the previous section: we have indeed found that as few as five steps are required to satisfy the condition $|z - w_s| < 10^{-5}$.

The calculation of the flow over a wing at zero yaw requires that we find that value of the determining parameter Δ for which the construction converges to $w = z = 0$ (to the accuracy required). This we have done by the method of false position, and have found that at most four calculations were needed to locate the convergence point to within a distance of 0·1 % of Ω (the reduced span) from the wing centre-line.

In figures 3(*a*) and (*b*) we show respectively the spanwise distribution of the pressure correction term p and the shock shape $y_s(z)$ for plane delta wings with reduced spans given by $\Omega = 2{\cdot}5$, 3 and 4. From these and other calculations, we have prepared the curve for Messiter's normal-force correction coefficient $F(\Omega)$, shown in figure 3(*c*). In his paper Messiter (1963) showed that, in the thin-shock-layer approximation, the normal-force coefficient for a delta wing in hypersonic flow (assuming the pressure on the leeward side to be negligible) could be expressed as the sum of the Newtonian value and a correction term of order ϵ, which depends only on the reduced span Ω of the wing:

$$C_N = 2\sin^2\alpha + 2/\gamma M_\infty^2 + \epsilon \sin^2\alpha F(\Omega),$$

where
$$F(\Omega) = \frac{2}{\Omega}\int_0^\Omega p(0, z; \Omega)\, dz.$$

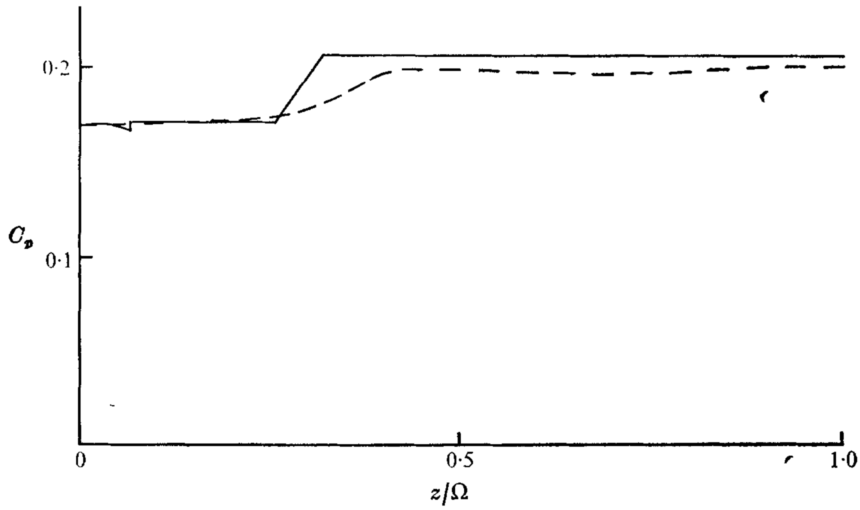


FIGURE 4. Comparison of predicted pressure distributions. $M = 5{\cdot}08$, $\alpha = 14°$, $\Lambda = 50°$. ——, this paper; - - - - - -, Klunker *et al.* (1971).

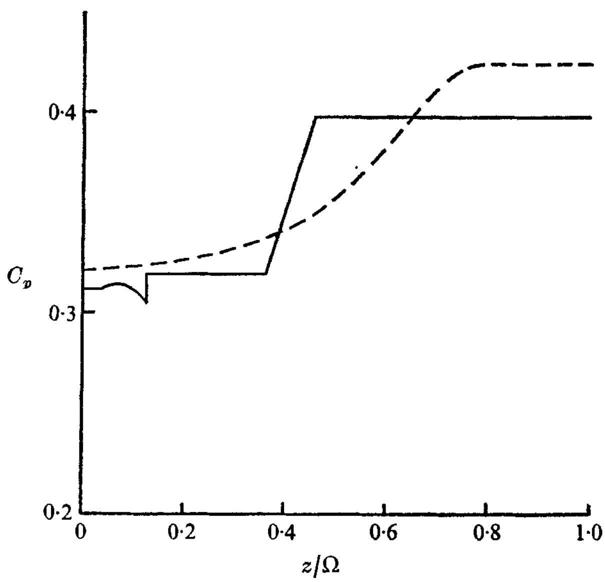


FIGURE 5. Comparison of predicted pressure distributions for conditions close to shock detachment. $M_\infty = 4$, $\alpha = 20°$, $\Lambda = 50°$. ——, this paper; - - - - - -, Klunker *et al.* (1971).

The curve which has been derived from our present calculations is indistinguishable from that obtained by one of us (Woods 1970) using an invalid flow construction.

In figure 4 we compare the pressure distribution predicted by thin-shock-layer theory with a calculation performed using the method of lines by Klunker *et al.* (1971) for a plane delta of 50° sweep-back at 14° incidence to flow at $M = 5{\cdot}08$. The latter appears to be representative of the most accurate numerical cal-

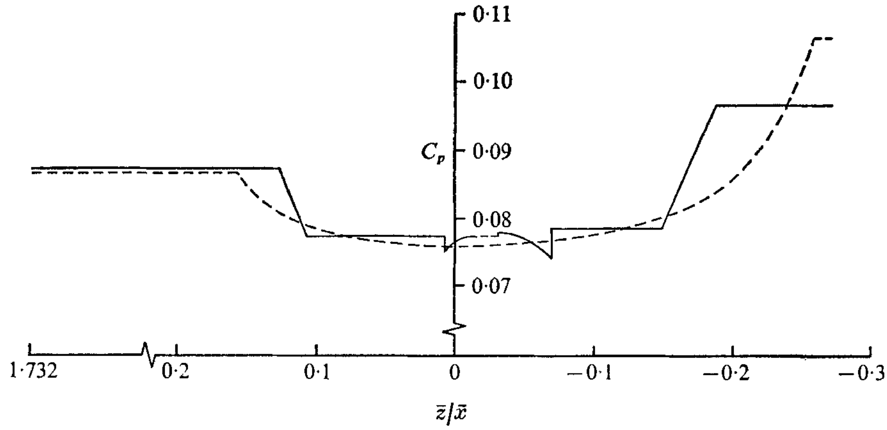


FIGURE 6. Comparison of predicted pressure distributions at yaw. $M_\infty = 10$, $\alpha = 10°$, $\beta = 22 \cdot 5°$, $\Lambda = 52 \cdot 5°$. ——, this paper; - - - - - -, Hui (1973).

culations available, although for our purpose the Mach number is somewhat low; the value of the 'small' parameter ϵ for this case is 0·718. Evidently, since the overall agreement is rather good, some of the errors inherent in thin-shock-layer theory have cancelled each other fortuitously for these flow conditions. This is not however the case for flow conditions close to shock detachment, where the differences are more substantial. This is illustrated in figure 5, where we compare the pressure distribution obtained by Klunker *et al.* for the same wing at 20° incidence to flow at $M = 4$ with that given by our solution. Whereas in the 'exact' solution the extent of the non-uniform central flow region is almost 75 % of the span, in the thin-shock-layer approximation it is less than 50 %.

We have been able to adapt the Hayes & Probstein solution to cases of asymmetric flow, on a delta wing at yaw, with the shock wave attached at both leading edges. This is done simply by treating the yawed delta as equivalent to two delta wings with different spans (i.e. values of Ω) and calculating flows for each, such that, if one flow converges to the station z_f at which the ordinate of the shock wave is y_{sf}, the other flow converges to $-z_f$, with the same final shock wave ordinate y_{sf}. Thus for an asymmetric flow of this sort, two separate values of Δ must be found, and the method of false position in this case is accordingly more tedious. We have illustrated this procedure (figure 6) by computing by our method a case treated by Hui (1973) using the improved linearized method which we have described in the introduction. This is for a delta wing of sweep angle 52·5° yawed by 22·5° and at an incidence of 10° to an $M = 10$ flow.† Because one edge is close to detachment, the pronounced departure of the thin-shock-layer result from the exact result which we remarked on above is again evident. It should be recalled that although Hui uses a linearized theory, he has adapted it such that the positions of the Mach cone and the uniform solution outboard of it are both exact.

† It is clear, incidentally, that the figure from which we have taken Hui's result (his figure 5) should bear the legend of his figure 6.

In all the calculations which we have presented in this section, the 'small' parameter ϵ has been numerically quite large. Indeed, it is probably the most questionable feature of this approximation that, for $\gamma = 1\cdot 4$, a parameter which is supposed to be vanishingly small is always greater than $\frac{1}{6}$.

5. Discussion and conclusions

In this paper we have presented a new general solution to the thin-shock-layer equations for conical flow over nearly plane wings. Using this solution, we have been able to construct a 'complete' realization of the solution for the attached-shock flow on a plane delta wing first outlined, and qualitatively described, by Hayes & Probstein (1966). This construction has been shown to be also applicable to asymmetric flows, as on a plane delta wing at an angle of yaw sufficiently small that the shock wave is still attached to both leading edges.

In the numerical results of our calculations, presented in § 4, the thin-shock-layer approximation appears to be only moderately accurate in predicting overall pressure levels. This shortcoming, and some suggestions for correcting it by semi-empirical means, have been discussed by Squire (1966, 1974). Of more fundamental interest is the other feature of the approximation which appears in these applications to attached-shock flows, namely the rich yield of anomalous and counter-intuitive phenomena predicted. In writing of the even greater variety of such phenomena predicted by Newtonian theory, Hayes & Probstein (1966) suggest that "apparently completely unrealistic anomalous phenomena may have significant vestigial counterparts in hypersonic gas flows. An apprecia-tion of the former is necessary for an understanding of the latter. Hence the anomalies of Newtonian theory are not to be avoided but rather to be sought out." The anomalies in our solution are undoubtedly less conspicuous that those of Newtonian theory; whether they have "significant vestigal counterparts" in real flows cannot be decided on the basis of the small number of comparisons with 'exact' numerical solutions which we have been able to make in § 4, since the latter are for flows at rather low Mach numbers for this purpose. Nor are we aware of any experimental results in which such effects as are seen in our solutions occur or might be expected. In connexion with the possibility of such effects being reproduced in finite-difference calculations based on the full equations, it should be pointed out that neither the method of 'optimization' (used by Babaev) nor the method of lines (used by Klunker *et al.*) seems adapted to the discovery of phenomena associated with sharp gradients of flow properties. If such phenomena exist, they are likely to be uncovered only by numerical procedures specifically devised not to suppress them, just as (for example) the method of Stocker & Mauger (1962) for calculating supersonic flow past cones of general cross-section was devised to prevent suppression of the vortical layer.

To conclude, we remark that some progress has been made towards the application of the thin-shock-layer approximation to calculate attached-shock flows on non-planar wings, and we hope to devote another paper to this topic. However, even greater difficulties arise in this case, of which we shall mention only what seems to be the most serious. It is this: for a non-planar wing there does not necessarily exist the equivalent of the 'simple wave' solution for the plane wing, so that the possibility of transition from one solution to another, which we exploited in the flow construction given above, is not open to us. The only arbitrary element which we have at our disposal is a super- to subcritical jump; and once this is introduced, an even more varied crop of anomalous features results, including a counterpart to the Newtonian discontinuity surface, with consequent complications in calculation.

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